Outline

- Stable lin system with vanishing perturbation (Sec 4.3)
- exponential stability (thm 4.10)
- Converse Lyapunov thm
- stable nonlinear sys with vanishing perturbation.

Consider

$$\dot{x} = Ax + g(x)$$

- Assume A is Hurwitz
- Assume $g(x)$ is a small perturbation such that

$$g(0) = 0 \quad \text{and} \quad \lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0$$

for example

$$g(x) = x\sqrt{x}$$

then $\frac{|g(x)|}{|x|} = \sqrt{x} \to 0$ as $x \to 0$

- Then, what can we say about stability?

- Because A is Hurwitz, for all p.d matrix Q there exists p.d matrix P that solves

$$PA + A^T P = -Q$$

- choose $Q = I$. Define Lyp. funct.

$$V(x) = x^T P x$$

p.d. ✓
radially unbounded ✓

- Then,

$$\begin{aligned}
\dot{V}(x) &= 2x^T P (Ax + g(x)) \\
&= x^T (PA + A^T P) + 2x^T P g(x) \\
&= -\underbrace{x^T Q x}_{-\|x\|^2} + 2x^T P g(x) \\
&\leq -\|x\|^2 + 2\|x\| \, \|P\| \, \|g(x)\|
\end{aligned}$$

- Because $\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0 \Rightarrow \forall \varepsilon > 0, \exists \delta$ s.t. $\|g(x)\| \leq \varepsilon \|x\|$ if $\|x\| \leq \delta$

- Therefore,

$$\dot{V}(x) \leq -\|x\|^2 + 2\varepsilon\|P\|\,\|x\|^2$$
$$= -(1 - 2\varepsilon\|P\|)\,\|x\|^2$$

- We are free to choose ε.

Let $\varepsilon = \frac{1}{4}\|P\|$ so that

$$\dot{V}(x) \leq -\frac{1}{2}\|x\|^2 \quad \text{for all } \|x\| \leq \delta$$

$$\Longrightarrow \boxed{x = 0 \text{ is AS}}$$

Remark:

- For any C^1 function $f(x)$, the linearization error $g(x) = f(x) - f(0) - \frac{\partial f}{\partial x}(0)\, x$ satisfies the property $\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0$

(see page (138)

- This is used to prove stability from linearization (thm 4.7)

- what can we say about rate of convergence?

- we have

$$V(x) = x^T P x$$

$$\dot{V}(x) \leq -\frac{1}{2}\|x\|^2 \quad \forall x \in D$$

$$D = \{x \mid \|x\| \leq \delta\}$$

- we use the inequality

$$\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2$$

- Note that all eigenvalues of P is positive because P is p.d. matrix.

- Use the inequality to obtain rate of convergence.

$$\dot{V}(x) = -\frac{1}{2}\|x\|^2 \leq \frac{-1}{2\lambda_{max}(P)} V(x)$$

$$\forall x \in D$$

- Since system is stable, we can choose r small enough s.t. $\|X_0\| \le r \Rightarrow \|X(t)\| \in D \quad \forall t$

- Therefore,

$$\frac{d}{dt} V(X(t)) = \dot{V}(X(t)) \le -\frac{1}{2\lambda_{max}} V(X(t)) \quad \forall t$$

- By application of comparison lemma 3.4

$\dot{x} = f(x)$ $\quad x, y \in \mathbb{R}$
$\dot{y} \le f(y)$ $\quad x(0) = y(0)$

$\Rightarrow y(t) \le x(t)$

$$V(X(t)) \le e^{-\frac{1}{2\lambda_{max}} t} V(X(0)) \quad \forall t$$

if $\|X_0\| \le r$

$$\Rightarrow \quad \|X(t)\|^2 \le \frac{1}{\lambda_{min}} V(X(t))$$

$$\le \frac{1}{\lambda_{min}} e^{-\frac{1}{2\lambda_{max}} t} V(X(0))$$

$$\le \frac{\lambda_{max}}{\lambda_{min}} e^{-\frac{1}{2\lambda_{max}} t} \|X(0)\|^2$$

if $\|X(0)\| \le r$

- We say $X = 0$ is exponential stable.

Def : (page 150)

- $x=0$ is exponentially stable if there exists positive constants $r, c, \lambda > 0$ s.t.

$$\|x(t)\| \leq c\|x(0)\| e^{-\lambda t}, \quad \forall \|x(0)\| \leq r$$

- $x=0$ is globally exponentially stable if

$$\|x(t)\| \leq c\|x(0)\| e^{-\lambda t} \quad \forall x_0$$

Thm : (thm 4.10)

- let $x=0$ be eqlb. for $\dot{x}=f(x)$
- let V be a C^1 function s.t.

$$K_1\|x\|^2 \leq V(x) \leq K_2\|x\|^2 \quad \forall x \in D$$

and

$$\dot{V}(x) \leq -K_3\|x\|^2 \quad \forall x \in \underset{\sim}{D}$$

open set containing $x=0$

Then,

$x=0$ is exponentially stable

- and $\|x(t)\| \le \sqrt{\frac{k_2}{k_1}}\,\|x(0)\|\, e^{-\frac{k_3}{2k_2}t}$

- If $D = \mathbb{R}^n \Longrightarrow$ globally exp. stable

Remark:

- $V(x) \ge k_1 \|x\|^2 \quad \forall x$ implies

  V is radially unbounded

- the power 2 can be changed to any positive constant.

Examples:

- Consider $\dot{x} = -x + x^3$

  and Lyp func. $V(x) = x^2$

- $x^2 \le V(x) \le x^2$ $\quad k_1 = k_2 = 1$ in thm.

$$\Longrightarrow \quad \dot{V}(x) = 2x(-x + x^3)$$

$$= -2x^2(1 - x^2)$$

$$\le -\underbrace{2(1-r^2)}_{k_3} x^2 \quad \text{if } |x| \le r < 1$$

$\Longrightarrow$ $x = 0$ is exponentially stable.

and $\|x(t)\| \leq \|x(0)\| e^{-(1-r^2)t}$ if $\|x(0)\| < r$

- Now if $\dot{x} = -x - x^3$

$$\Rightarrow \quad \dot{V}(x) = -2x^2 - 2x^4$$

$$\leq -2x^2 \qquad \forall x$$

$\Rightarrow$ globally exp stable

$$\|x(t)\| \leq \|x(0)\| e^{-t} \qquad \forall x_0$$

We saw that

$$\dot{x} = Ax + g(x) \quad \text{with} \quad \lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0$$

is exponentially stable if $\dot{x} = Ax$ is exponentially stable (or A is Hurwitz).

- Can we say the same thing for

$\dot{x} = f(x) + g(x)$ if $\dot{x} = f(x)$ is exp stable?

- like the linear case, we need to use a Lyapunov function for $\dot{x} = f(x)$

Converse Lyapunov thm: (thm. 4.14)

- Suppose, $x=0$ is exp stable for $\dot{x} = f(x)$ where f is C^1

$$\|x(t)\| \leq C e^{-\lambda t} \|x(0)\| \quad \forall x_0 \in D_0$$

- Then, there exists a Lyapunov function $V: D_0 \to \mathbb{R}$ such that

- $k_1 \|x\|^2 \leq V(x) \leq k_2 \|x\|^2$ $\quad \forall x \in D_0$
- $\dot{V}(x) \leq -k_3 \|x\|^2$
- $\left\| \frac{\partial V}{\partial x}(x) \right\| \leq k_4 \|x\|$

for some positive constants k_1, k_2, k_3, k_4.

- if global exp stable, then $D_0 = \mathbb{R}^n$

- We can use this result to show

$$\dot{x} = f(x) + g(x) \quad \text{with} \quad \frac{\|g(x)\|}{\|x\|} \to 0$$

is exp stable if $\dot{x} = f(x)$ is exp stable

Sec 9.1

- Use the Lyapunov funct. in thm.

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)\,(f(x) + g(x))$$

$$\leq -K_3 \|x\|^2 + \frac{\partial V}{\partial x} g(x)$$

$$\leq -K_3 \|x\|^2 + \varepsilon K_4 \|x\| \, \|g(x)\|$$

- for all $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\|g(x)\| \leq \varepsilon \|x\|$ if $\|x\| \leq \delta$

$$\Rightarrow \quad \dot{V}(x) \leq -\|x\|^2 (K_3 - \varepsilon K_4)$$

- let $\varepsilon = \dfrac{K_3}{2K_4}$ $\Rightarrow$ $\dot{V}(x) \leq -\dfrac{K_3}{2}\|x\|^2$ $\Rightarrow$ exp stable